

Depth reduction of a class of Witten zeta functions

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Abstract

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We show that if a, b, c, d, f are positive integers such that $a + b + c + d + f$ is even, then the Witten zeta value $\zeta_{\mathfrak{sl}(4)}(a, b, c, d, 0, f)$ is expressible in terms of Witten zeta functions with fewer arguments.

1 Introduction

Let \mathbf{N} be the set of positive integers, \mathbf{Q} the field of rational numbers, \mathbf{C} the field of complex numbers.

For any semisimple Lie algebra \mathfrak{g} , the Witten zeta function(cf. [6]) is defined by

$$\zeta_{\mathfrak{g}}(s) = \sum_{\rho} (\dim \rho)^{-s},$$

where $s \in \mathbf{C}$ and ρ runs over all finite dimensional irreducible representations of \mathfrak{g} . In order to calculate the volumes of certain moduli space, Witten [8] introduced the values $\zeta_{\mathfrak{g}}(2k)$ for $k \in \mathbf{N}$ and showed that $\pi^{-2kl} \zeta_{\mathfrak{g}}(2k) \in \mathbf{Q}$, where l is the number of positive roots of \mathfrak{g} .

For positive integer r , Matsumoto and Tsumura [6] defined a multi-variate extension, called the Witten multiple zeta-function associated with $\mathfrak{sl}(r+1)$, by

$$\zeta_{\mathfrak{sl}(r+1)}(\mathbf{s}) = \sum_{m_1, \dots, m_r=1}^{\infty} \prod_{j=1}^r \prod_{k=1}^{r-j+1} \left(\sum_{v=k}^{j+k-1} m_v \right)^{-s_{j,k}} \quad (1)$$

where

$$\mathbf{s} = (s_{j,k})_{1 \leq j \leq r; 1 \leq k \leq r-j+1} \in \mathbf{C}^{r(r+1)/2}, \quad \Re(s_{j,k}) > 1.$$

In particular ([6], section 2, Proposition 2.1), if $m \in \mathbf{N}$ we denote

$$\zeta_{\mathfrak{sl}(r+1)}(2m) := \prod_{1 \leq j < k \leq r+1} (k-j) \zeta_{\mathfrak{sl}(r+1)}(\underbrace{2m, \dots, 2m}_{r(r+1)/2}).$$

As in [1], given the Witten multiple zeta-function (1), we define the *depth* to be r . Further, if the zeta functions y_1, \dots, y_k have depth r_1, \dots, r_k respectively, then for $a_1, \dots, a_k \in \mathbf{C}$, we define the *depth* of $a_1 y_1 + \dots + a_k y_k$ to be $\max\{r_i : 1 \leq i \leq k\}$. We would like to know which sums can be expressed in terms of lower depth sums. When a sum can be so expressed, we say it is *reducible*.

An explicit evaluation for $\zeta_{\mathfrak{sl}(3)}(2m)$ ($m \in \mathbf{N}$) was independently discovered by D. Zagier, S. Garoufalidis, and L. Weinstein (see [9, page 506]). In [4],

Gunnells and Sczech provided a generalization of the continued-fraction algorithm to compute high-dimensional Dedekind sums. As examples, they gave explicit evaluations of $\zeta_{\text{sl}(3)}(2m)$ and $\zeta_{\text{sl}(4)}(2m)$. Matsumoto and Tsumura [6] considered functional relations for Witten multiple zeta-functions, and found that

$$\begin{aligned} & (-1)^a \zeta_{\text{sl}(4)}(s_1, s_2, a, s_3, 0, b) + (-1)^b \zeta_{\text{sl}(4)}(s_1, s_2, b, s_3, 0, a) \\ & + \zeta_{\text{sl}(4)}(a, 0, s_2, s_1, b, s_3) + \zeta_{\text{sl}(4)}(b, 0, s_1, s_2, a, s_3) \end{aligned} \quad (2)$$

is reducible for any $a, b \in \mathbf{N}$ and $s_1, s_2, s_3 \in \mathbf{C}$.

In this paper, we provide a combinatorial method which gives a simpler formula for the quantity (2). Furthermore, we show that if a, b, c, d, f are positive integers such that $a + b + c + d + f$ is even, then $\zeta_{\text{sl}(4)}(a, b, c, d, 0, f)$ is reducible.

2 Functional relation

Lemma 2.1. *If the function $F : \mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0} \times \mathbf{C} \rightarrow \mathbf{C}$ has the property that there exist $p, q \in \mathbf{C}$ such that for every $a, b \in \mathbf{N}$ and every $s \in \mathbf{C}$ the relation*

$$F(a, b, s) = pF(a-1, b, s+1) + qF(a, b-1, s+1)$$

holds, then for every $a, b \in \mathbf{N}$ and every $s \in \mathbf{C}$,

$$\begin{aligned} F(a, b, s) &= \sum_{j=1}^b p^a q^{b-j} \binom{a+b-j-1}{a-1} F(0, j, a+b+s-j) \\ &+ \sum_{j=1}^a p^{a-j} q^b \binom{a+b-j-1}{b-1} F(j, 0, a+b+s-j). \end{aligned} \quad (3)$$

Proof. It's easy to prove Lemma 2.1 by induction. □

The Euler sum of depth r and weight w is a multiple series of the form

$$\zeta(s_1, \dots, s_r) := \sum_{n_1 > \dots > n_r > 0} \prod_{j=1}^r n_j^{-s_j}, \quad (4)$$

with *weight* $w := s_1 + \dots + s_r$. Now let's recall the following result concerning the reduction on the triple Euler sums.

Lemma 2.2 (Borwein-Girgensohn [2]). *Let a, b, c be positive integers. If $a + b + c$ is even or less than or equal to 10, then $\zeta(a, b, c)$ can be expressed as a rational linear combination of products of single and double Euler sums of weight $a + b + c$.*

Lemma 2.3 (Huard-Williams-Zhang [5]). *If a, b, c be positive integers, then*

$$\zeta_{\text{sl}(3)}(a, b, c) = \left\{ \sum_{j=1}^a \binom{a+b-j-1}{b-1} + \sum_{j=1}^b \binom{a+b-j-1}{a-1} \right\} \zeta(a+b+c-j, j). \quad (5)$$

Moreover, $\zeta_{\text{sl}(3)}(a, b, c)$ can be explicitly evaluated in terms of the values of Riemann zeta functions when $a + b + c$ is odd.

Theorem 2.1. *If $a, b \in \mathbf{N}$, then*

$$\begin{aligned} & (-1)^a \zeta_{\text{sl}(4)}(s_1, s_2, a, s_3, 0, b) + (-1)^b \zeta_{\text{sl}(4)}(s_1, s_2, b, s_3, 0, a) \\ & \quad + \zeta_{\text{sl}(4)}(a, 0, s_2, s_1, b, s_3) + \zeta_{\text{sl}(4)}(b, 0, s_1, s_2, a, s_3) \\ &= \sum_{i=1}^{\max(a,b)} \left\{ \binom{a+b-i-1}{a-1} + \binom{a+b-i-1}{b-1} \right\} (-1)^i \zeta(i) \\ & \quad \times \zeta_{\text{sl}(3)}(s_1, s_2, s_3 + a + b - i) \\ & \quad + \sum_{i=1}^a \binom{a+b-i-1}{b-1} \left\{ \zeta(i) \zeta_{\text{sl}(3)}(s_1, s_2, s_3 + a + b - i) \right. \\ & \quad \left. - \zeta_{\text{sl}(3)}(s_1 + i, s_2, s_3 + a + b - i) - \zeta_{\text{sl}(3)}(s_1, s_2, s_3 + a + b) \right\} \\ & \quad + \sum_{i=1}^b \binom{a+b-i-1}{a-1} \left\{ \zeta(i) \zeta_{\text{sl}(3)}(s_1, s_2, s_3 + a + b - i) \right. \\ & \quad \left. - \zeta_{\text{sl}(3)}(s_2 + i, s_1, s_3 + a + b - i) - \zeta_{\text{sl}(3)}(s_1, s_2, s_3 + a + b) \right\}. \end{aligned} \quad (6)$$

Proof. From the definition (1) of the Witten multiple zeta-function, we have

$$\zeta_{\text{sl}(4)}(s_1, s_2, s_3, s_4, s_5, s_6) = \zeta_{\text{sl}(4)}(s_3, s_2, s_1, s_5, s_4, s_6). \quad (7)$$

Next, for any $a, b \in \mathbf{N}$ and $s_1, s_2, s_3 \in \mathbf{C}$, since

$$\begin{aligned}\zeta_{\mathfrak{sl}(4)}(s_1, s_2, a, s_3, 0, b) &= \zeta_{\mathfrak{sl}(4)}(s_1, s_2, a, s_3 + 1, 0, b - 1) \\ &\quad - \zeta_{\mathfrak{sl}(4)}(s_1, s_2, a - 1, s_3 + 1, 0, b),\end{aligned}$$

by Lemma 2.1, we have

$$\begin{aligned}\zeta_{\mathfrak{sl}(4)}(s_1, s_2, a, s_3, 0, b) &= \sum_{i=1}^a \binom{a+b-i-1}{b-1} (-1)^{a+i} \zeta_{\mathfrak{sl}(4)}(s_1, s_2, i, s_3 + a + b - i, 0, 0) \\ &\quad + \sum_{i=1}^b \binom{a+b-i-1}{a-1} (-1)^a \zeta_{\mathfrak{sl}(4)}(s_1, s_2, 0, s_3 + a + b - i, 0, i).\end{aligned}\tag{8}$$

Similarly, we have

$$\begin{aligned}\zeta_{\mathfrak{sl}(4)}(s_1, s_2, b, s_3, 0, a) &= \sum_{i=1}^b \binom{a+b-i-1}{a-1} (-1)^{b+i} \zeta_{\mathfrak{sl}(4)}(s_1, s_2, i, s_3 + a + b - i, 0, 0) \\ &\quad + \sum_{i=1}^a \binom{a+b-i-1}{b-1} (-1)^b \zeta_{\mathfrak{sl}(4)}(s_1, s_2, 0, s_3 + a + b - i, 0, i),\end{aligned}\tag{9}$$

$$\begin{aligned}\zeta_{\mathfrak{sl}(4)}(a, 0, s_2, s_1, b, s_3) &= \sum_{i=1}^a \binom{a+b-i-1}{b-1} \zeta_{\mathfrak{sl}(4)}(i, 0, s_2, s_1, 0, s_3 + a + b - i) \\ &\quad + \sum_{i=1}^b \binom{a+b-i-1}{a-1} \zeta_{\mathfrak{sl}(4)}(0, 0, s_2, s_1, i, s_3 + a + b - i),\end{aligned}\tag{10}$$

and

$$\begin{aligned}\zeta_{\mathfrak{sl}(4)}(b, 0, s_1, s_2, a, s_3) &= \sum_{i=1}^b \binom{a+b-i-1}{a-1} \zeta_{\mathfrak{sl}(4)}(i, 0, s_1, s_2, 0, s_3 + a + b - i) \\ &\quad + \sum_{i=1}^a \binom{a+b-i-1}{b-1} \zeta_{\mathfrak{sl}(4)}(0, 0, s_1, s_2, i, s_3 + a + b - i).\end{aligned}\tag{11}$$

Since

$$\zeta_{\mathfrak{sl}(4)}(a, b, c, d, 0, 0) = \zeta(c) \zeta_{\mathfrak{sl}(3)}(a, b, d), \quad (12)$$

$$\begin{aligned} \zeta_{\mathfrak{sl}(4)}(a, b, 0, c, 0, d) &= \sum_{\substack{n_1, n_2=1 \\ v > n_1 + n_2}} \frac{1}{v^d n_1^a n_2^b (n_1 + n_2)^c} \\ &= \sum_{\substack{n_1, n_2=1 \\ v > n_1 + n_2}} \frac{1}{v^d n_1^b n_2^a (n_1 + n_2)^c}, \end{aligned} \quad (13)$$

$$\zeta_{\mathfrak{sl}(4)}(a, 0, b, c, 0, d) = \sum_{\substack{n_1, n_2=1 \\ v < n_1}} \frac{1}{v^a n_1^c n_2^b (n_1 + n_2)^d}, \quad (14)$$

$$\zeta_{\mathfrak{sl}(4)}(0, 0, a, b, c, d) = \sum_{\substack{n_1, n_2=1 \\ n_1 + n_2 > v > n_1}} \frac{1}{v^c n_1^a n_2^b (n_1 + n_2)^d}, \quad (15)$$

we find that

$$\begin{aligned} &\zeta_{\mathfrak{sl}(4)}(s_1, s_2, 0, s_3 + a + b - i, 0, i) + \zeta_{\mathfrak{sl}(4)}(i, 0, s_2, s_1, 0, s_3 + a + b - i) \\ &\quad + \zeta_{\mathfrak{sl}(4)}(0, 0, s_1, s_2, i, s_3 + a + b - i) \\ &= \zeta(i) \zeta_{\mathfrak{sl}(3)}(s_1, s_2, s_3 + a + b - i) - \zeta_{\mathfrak{sl}(3)}(s_1 + i, s_2, s_3 + a + b - i) \\ &\quad - \zeta_{\mathfrak{sl}(3)}(s_1, s_2, s_3 + a + b) \end{aligned} \quad (16)$$

and

$$\begin{aligned} &\zeta_{\mathfrak{sl}(4)}(s_1, s_2, 0, s_3 + a + b - i, 0, i) + \zeta_{\mathfrak{sl}(4)}(i, 0, s_1, s_2, 0, s_3 + a + b - i) \\ &\quad + \zeta_{\mathfrak{sl}(4)}(0, 0, s_2, s_1, i, s_3 + a + b - i) \\ &= \zeta(i) \zeta_{\mathfrak{sl}(3)}(s_1, s_2, s_3 + a + b - i) - \zeta_{\mathfrak{sl}(3)}(s_2 + i, s_1, s_3 + a + b - i) \\ &\quad - \zeta_{\mathfrak{sl}(3)}(s_1, s_2, s_3 + a + b) \end{aligned} \quad (17)$$

Now combining equations (8-17), we complete the proof. \square

Lemma 2.4. *Every Witten multiple zeta value of the form $\zeta_{\mathfrak{sl}(4)}(a, b, 1, d, 0, 1)$ with $a, b, d \in \mathbf{N}$ can be expressed as a rational linear combination of products of single and double Euler sums when $a + b + d$ is even or $a + b + d \leq 8$.*

Proof.

$$\begin{aligned}\zeta_{\mathfrak{sl}(4)}(a, b, 1, d, 0, 1) &= \sum_{i=1}^a \binom{a+b-i-1}{b-1} \zeta_{\mathfrak{sl}(4)}(i, 0, 1, a+b+d-i, 0, 1) \\ &\quad + \sum_{i=1}^b \binom{a+b-i-1}{a-1} \zeta_{\mathfrak{sl}(4)}(0, i, 1, a+b+d-i, 0, 1).\end{aligned}\tag{18}$$

However, for any $a, d \in \mathbf{N}$,

$$\begin{aligned}\zeta_{\mathfrak{sl}(4)}(a, 0, 1, d, 0, 1) &= \zeta_{\mathfrak{sl}(4)}(0, a, 1, d, 0, 1) \\ &= \zeta_{\mathfrak{sl}(4)}(a, 0, 1, 0, 0, d+1) + \sum_{i=1}^d \zeta(d+2-i, i, a),\end{aligned}\tag{19}$$

and

$$\zeta_{\mathfrak{sl}(4)}(a, 0, 1, 0, 0, d+1) = \zeta(d+1, a, 1) + \sum_{i=1}^a \zeta(d+1, a+1-i, i).\tag{20}$$

We complete the proof by combining this with Lemma 2.2. \square

Theorem 2.2. *Every Witten multiple zeta value of the form $\zeta_{\mathfrak{sl}(4)}(a, b, c, d, 0, f)$ with $a, b, c, d, f \in \mathbf{N}$ can be expressed as a rational linear combination of products of single and double Euler sums when $a + b + c + d + f$ is even or $a + b + c + d + f \leq 10$.*

Proof. From Lemma 2.1, we see that

$$\begin{aligned}&\frac{1}{n_1^a n_2^b n_3^c (n_1 + n_2)^d (n_1 + n_2 + n_3)^f} \\ &= \sum_{i=1}^c \binom{c+f-i-1}{f-1} (-1)^{c+i} \frac{1}{n_1^a n_2^b n_3^i (n_1 + n_2)^{c+d+f-i}} \\ &\quad + \sum_{i=1}^f \binom{c+f-i-1}{c-1} (-1)^c \frac{1}{n_1^a n_2^b (n_1 + n_2)^{c+d+f-i} (n_1 + n_2 + n_3)^i}.\end{aligned}\tag{21}$$

Also

$$\begin{aligned}
& \frac{1}{n_1^a n_2^b (n_1 + n_2)^{c+d+f-i} (n_1 + n_2 + n_3)^i} \\
&= \sum_{j=1}^a \binom{a+b-j-1}{b-1} \frac{1}{n_1^j (n_1 + n_2)^{a+b+c+d+f-i-j} (n_1 + n_2 + n_3)^i} \\
&+ \sum_{j=1}^b \binom{a+b-j-1}{a-1} \frac{1}{n_2^j (n_1 + n_2)^{a+b+c+d+f-i-j} (n_1 + n_2 + n_3)^i}. \tag{22}
\end{aligned}$$

Now combine (20), (21) and Lemma 2.4 and sum over all ordered triples of positive integers (n_1, n_2, n_3) to obtain

$$\begin{aligned}
\zeta_{\text{sl}(4)}(a, b, c, d, 0, f) &= \sum_{i=2}^c \binom{c+f-i-1}{f-1} (-1)^{c+i} \zeta(i) \zeta_{\text{sl}(3)}(a, b, c+d+f-i) \\
&+ \sum_{i=2}^f \binom{c+f-i-1}{c-1} (-1)^c \left\{ \sum_{j=1}^a \binom{a+b+j-1}{b-1} \right. \\
&\quad \times \zeta(i, c+d+f+a+b-i-j, j) \\
&\quad \left. + \sum_{j=1}^b \binom{a+b+j-1}{a-1} \zeta(i, c+d+f+a+b-i-j, j) \right\} \\
&- (-1)^c \binom{c+f-2}{c-1} \zeta_{\text{sl}(4)}(a, b, 1, c+d+f-2, 0, 1). \tag{23}
\end{aligned}$$

By Lemmas 2.2, 2.3 and 2.4, we complete the proof. \square

Remark. When $d = 0$, the Witten zeta value $\zeta_{\text{sl}(4)}(a, b, c, 0, 0, f)$ can also be viewed as a Mordell-Tornheim sum with depth 3. The fact that every such sum can be expressed as a rational linear combination of products of single and double Euler sums when the weight $a + b + c + f$ is even has been shown in [7] and [3].

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